Clocks on the Moon and in Cislunar space: Lessons from GPS

Image credit: SpaceX/NASA

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National Institute of Standards and Technology
U.S. Department of Commerce

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Artemis: NASA

Artemis

Why We Are Going To The Moon

We're going back to the Moon for scientific discovery, economic benefits, and inspiration for a new generation of explorers: the Artemis Generation. While maintaining American leadership in exploration, we will build a global alliance and explore deep space for the benefit of all.

Learn More About the Moon

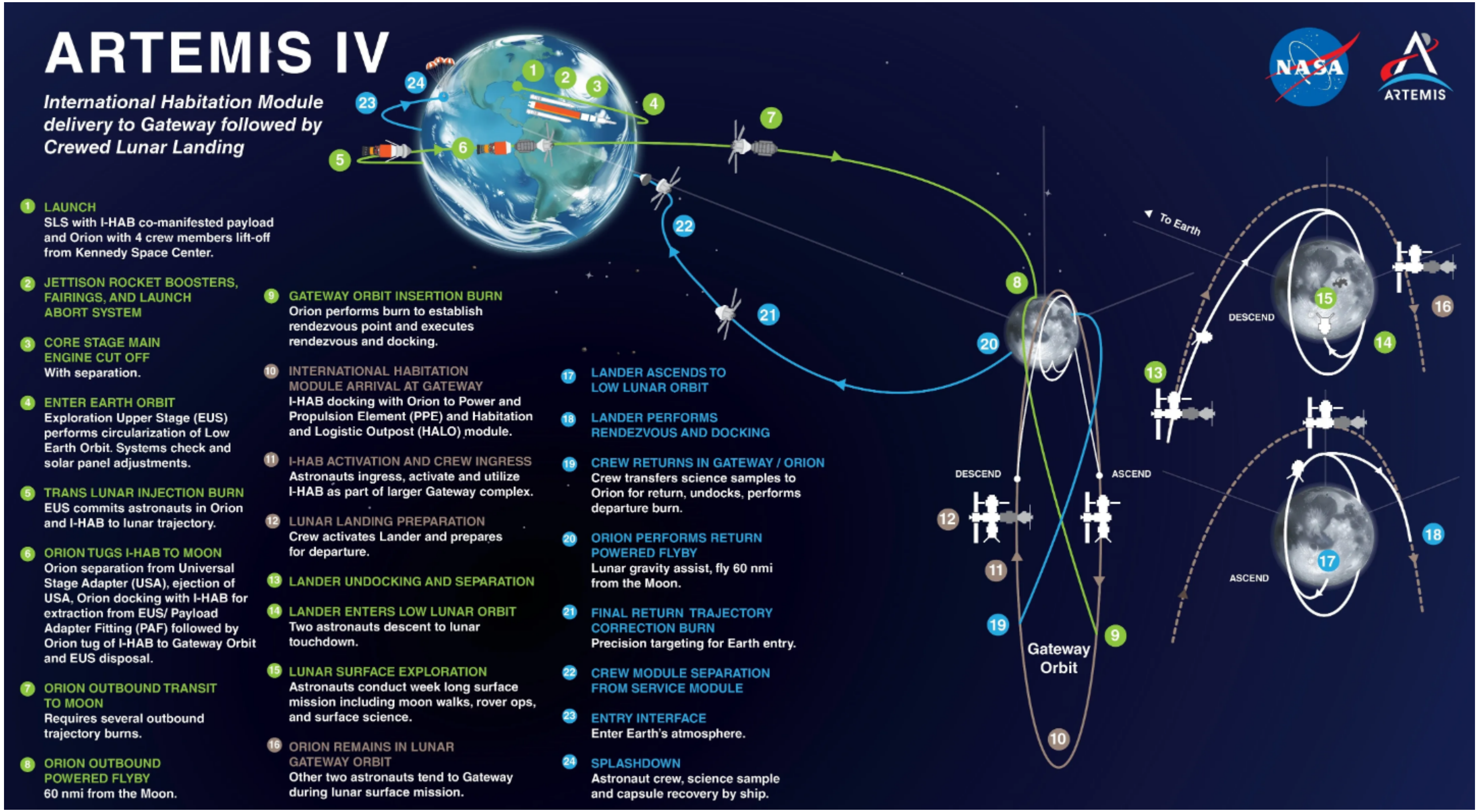
Waxing gibbous Moon at 11 days old.
Ernie Wright / NASA

Our success will change the world.

- In situ resource use
    - Low gravity
    - Low temperature
    - Almost ideal vacuum
- Biotechnology
    - Develop uniform crystals
    - Ideal for growing protein
    - Drugs & pharmaceuticals
- Navigation
    - Precision landing (~10m)
    - Path estimation

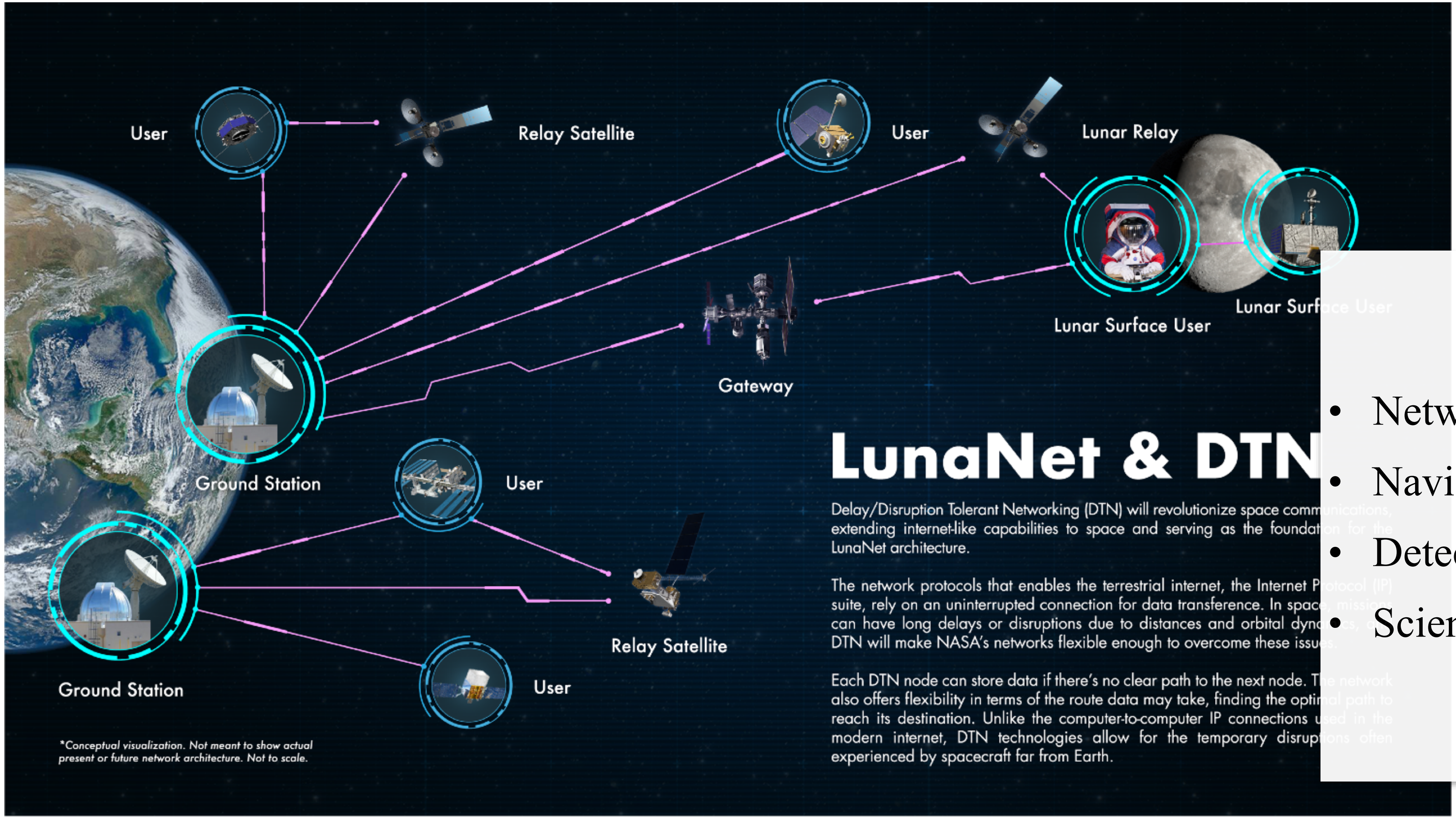
https://www.nasa.gov/specials/artemis/index.html

Artemis: NASA



https://www.nasa.gov/general/nasas-artemis-iv-building-first-lunar-space-station/

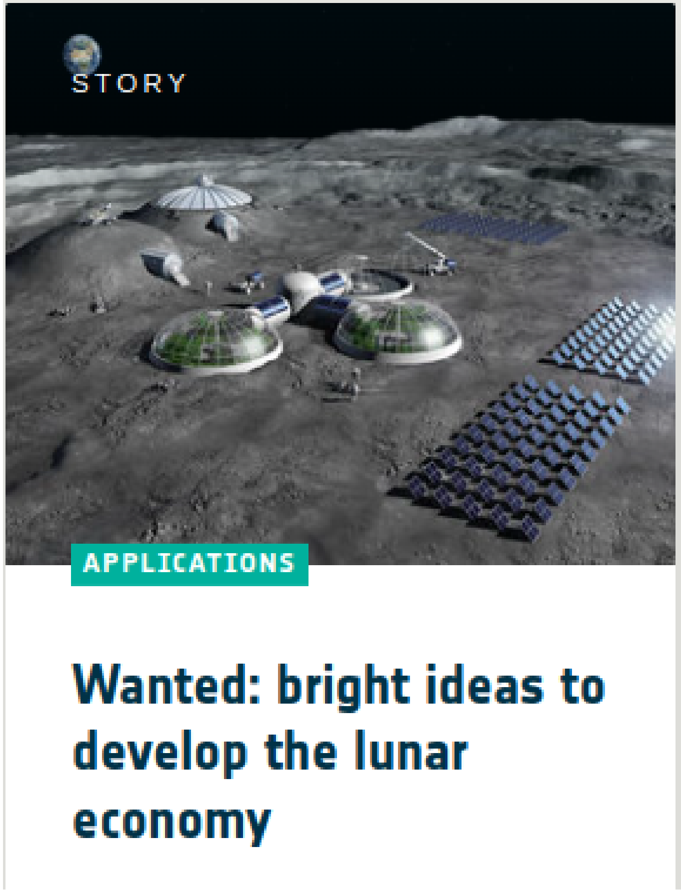
LunaNet:NASA

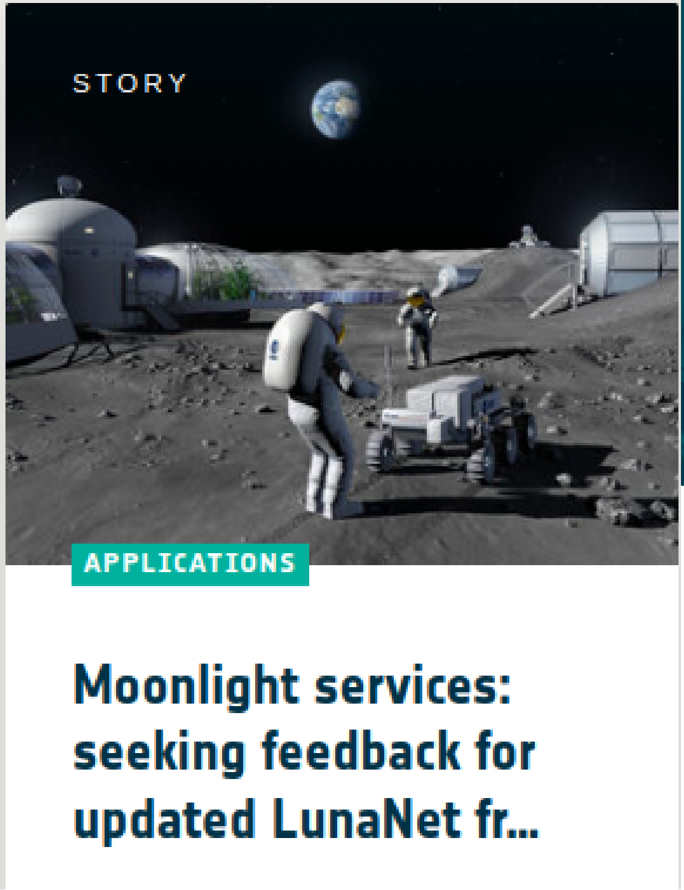


- Network (internet for the moon)
- Navigation
- Detection and information
- Science services

https://www.nasa.gov/humans-in-space/lunanet-empowering-artemis-with-communications-and-navigation-interoperability/

Moonlight: European Space Agency (ESA)

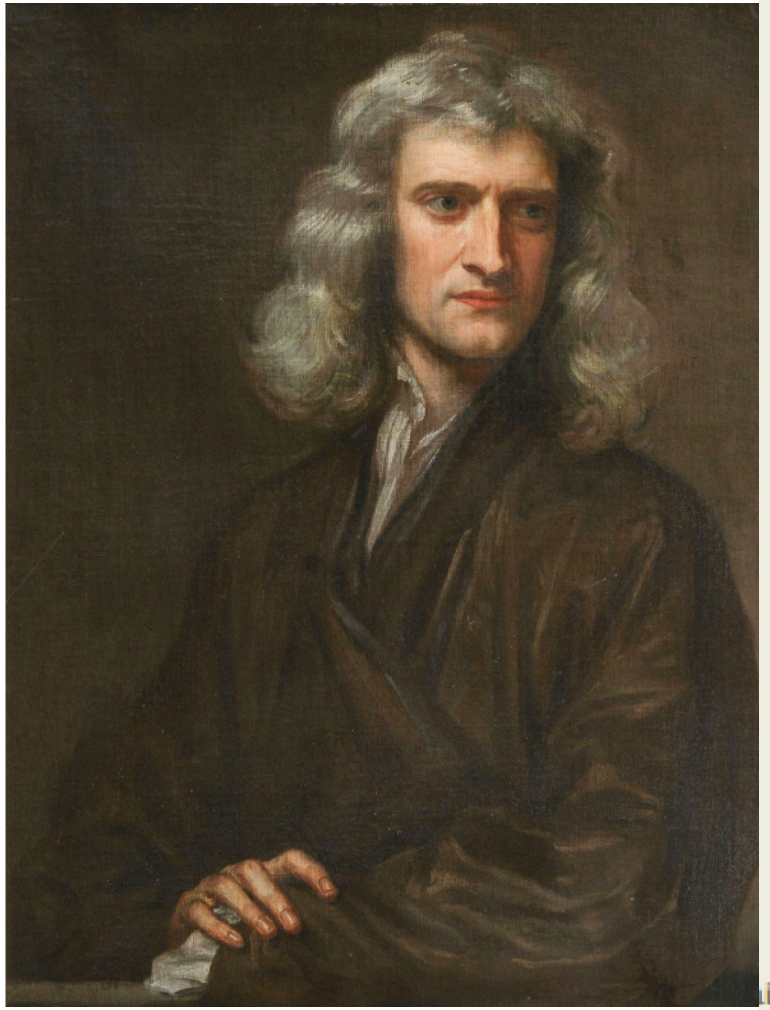




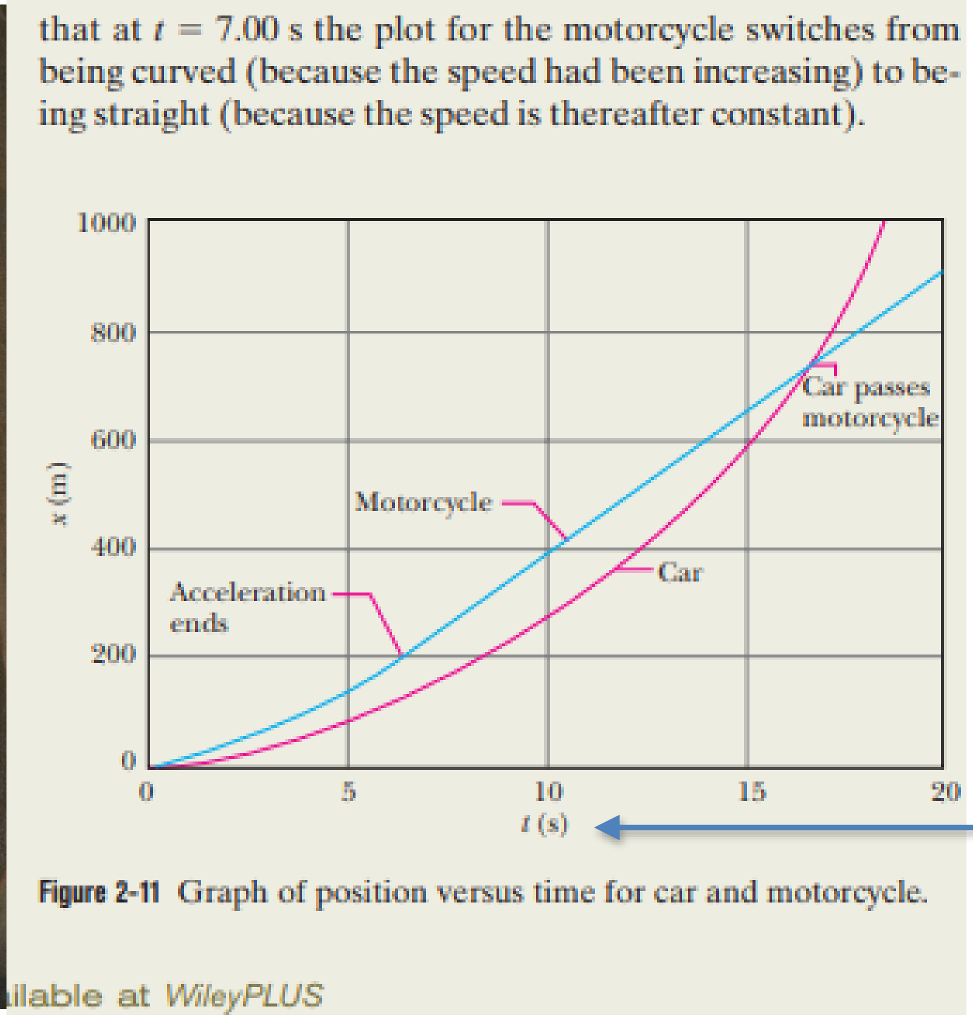
- First off-planet commercial telecoms and satellite navigation provider
- Planned operations on the South Pole
- ~5 satellites in lunar orbit, plus a relay
- ~15 hours of coverage per day

https://www.esa.int/ESA_Multimedia/Videos/2022/11/What_is_ESA_s_Moonlight_initiative

Time: From Newtonian to Einstein



Credit: Wikipedia



that at t = 7.00 s the plot for the motorcycle switches from being curved (because the speed had been increasing) to being straight (because the speed is thereafter constant).

Figure 2-11 Graph of position versus time for car and motorcycle.

ilable at *WileyPLUS*

Space and time are absolute.

Space extends to infinity in all three directions, and time is the same at every point in space at any given instant.

The rate of clocks is the same everywhere.

Coordinate time

Time: Einstein's theory of relativity

~~Space and time are absolute.~~

There's only spacetime. Every point in spacetime is an event.

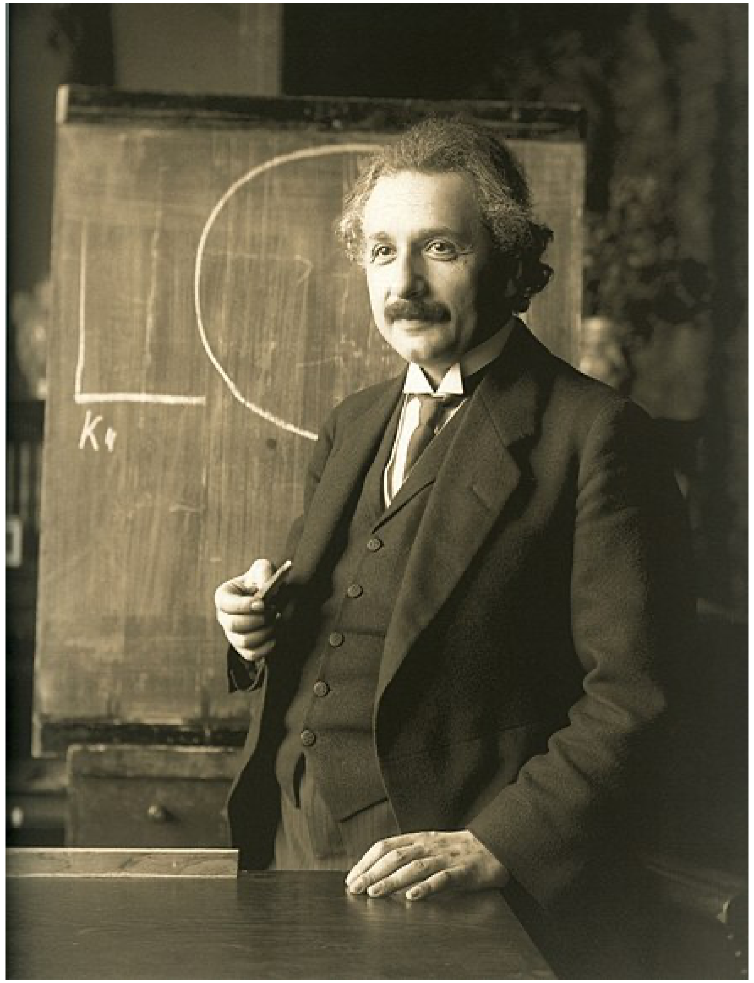
~~Space extends to infinity in all three directions, and time is the same at every point in space at any given instant.~~

Length and time intervals are relative. Clocks tell the proper time.

~~The rate of clocks is the same everywhere.~~

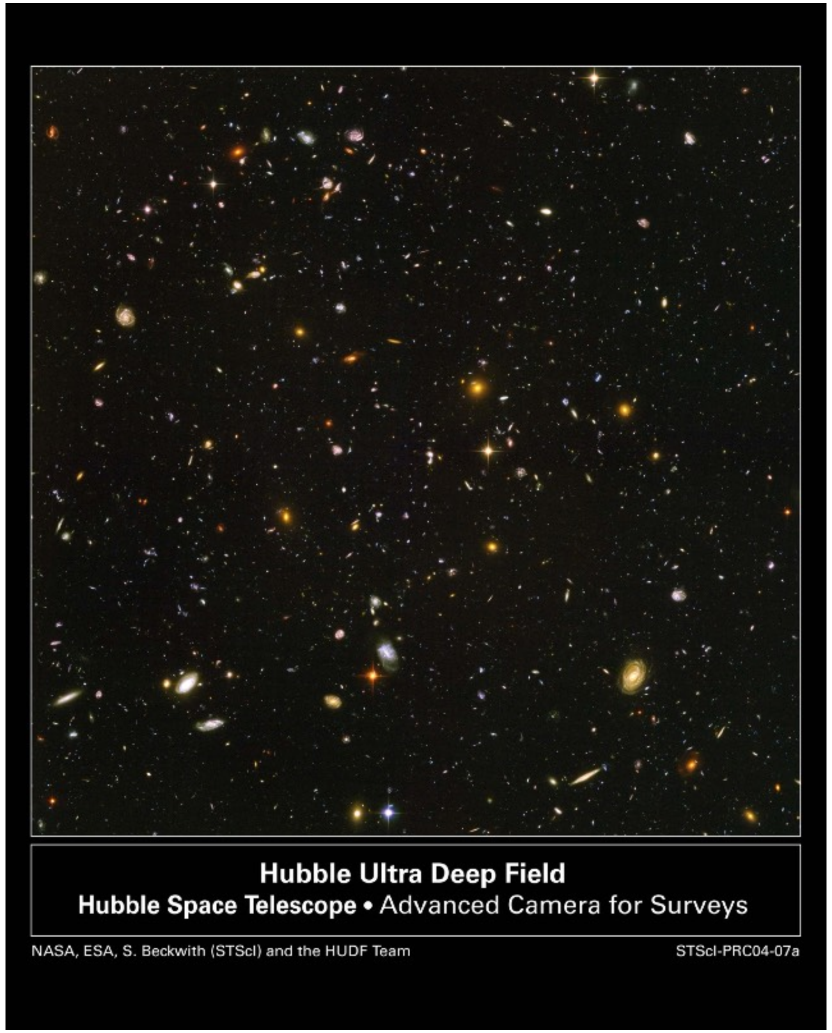
The motion of the clock and gravity at the location of the clocks determine the rate of clocks.

Newton's notion of space and time matches Einstein's only if the universe is empty and static.

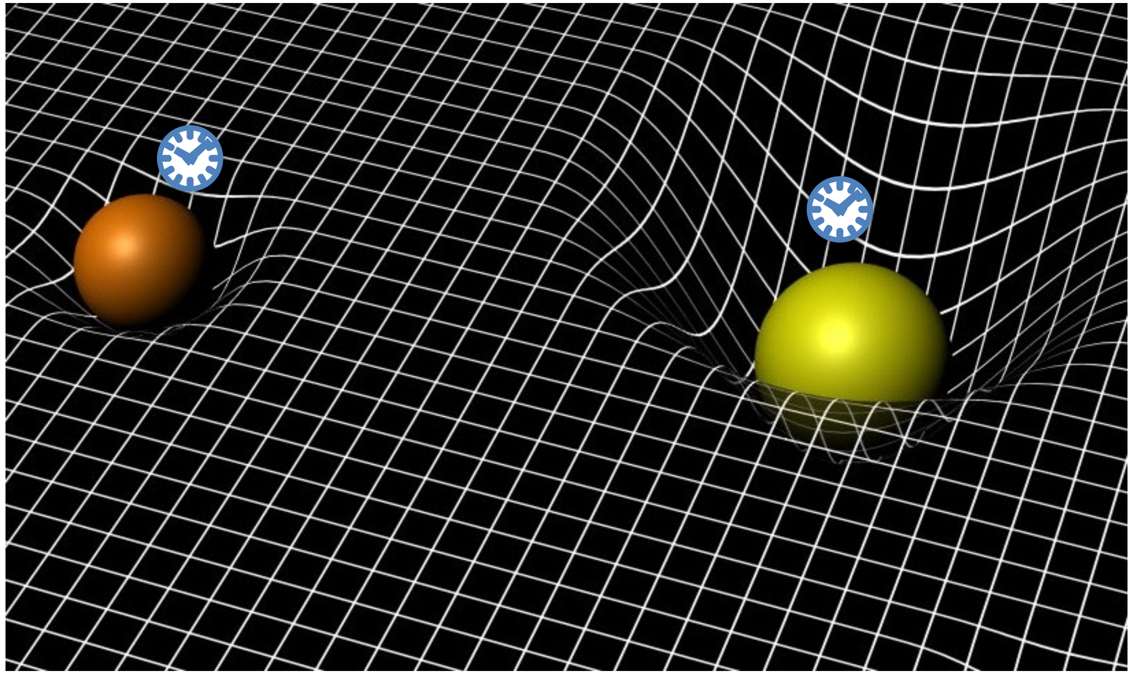


Credit: Wikipedia

Spacetime and Relativity



Our very beautiful universe is neither empty nor static.

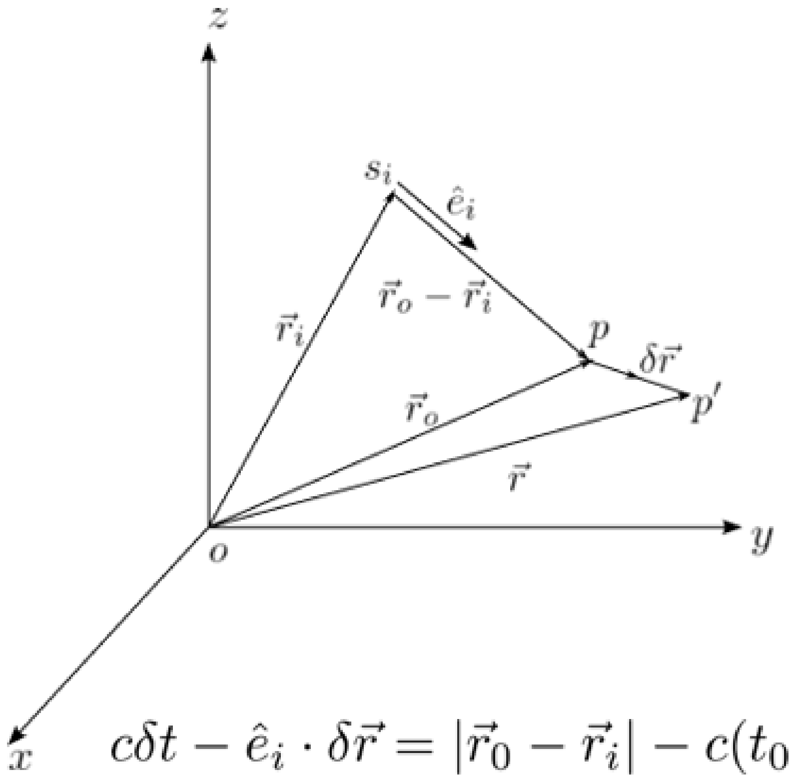


Credit: ESA

“Spacetime tells matter how to move, and matter tells spacetime how to curve.” —John Wheeler?

In practice, it means that the rate of clocks and length intervals determined by “light clocks” are determined by gravity and relative motion.

GPS and coordinate time



$$c\delta t - \hat{e}_i \cdot \delta\vec{r} = |\vec{r}_0 - \vec{r}_i| - c(t_0 - t_i)$$

$$GX \qquad\qquad Y$$

$$X = (G^T G)^{-1} G^T Y$$

$\vec{r}_i$ position vector of satellite

$\vec{r}_0$ initial guess for the receiver position

$\vec{r}$ computed position of the receiver

$t_i = \rho_R / c$, is the coordinate time at which satellite i emits a signal.

GPS time (UTC without leap seconds)

$$\hat{e}_i = \frac{\vec{r}_0 - \vec{r}_i}{|\vec{r}_0 - \vec{r}_i|}$$

$$c = 299792458.0\, m/s$$

$$X = \begin{pmatrix} c\delta t \\ \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

- converges in 6 iterations
- 4 to 11 satellites

GPS satellite and ISS comparison

GPS III, Credit: USAF

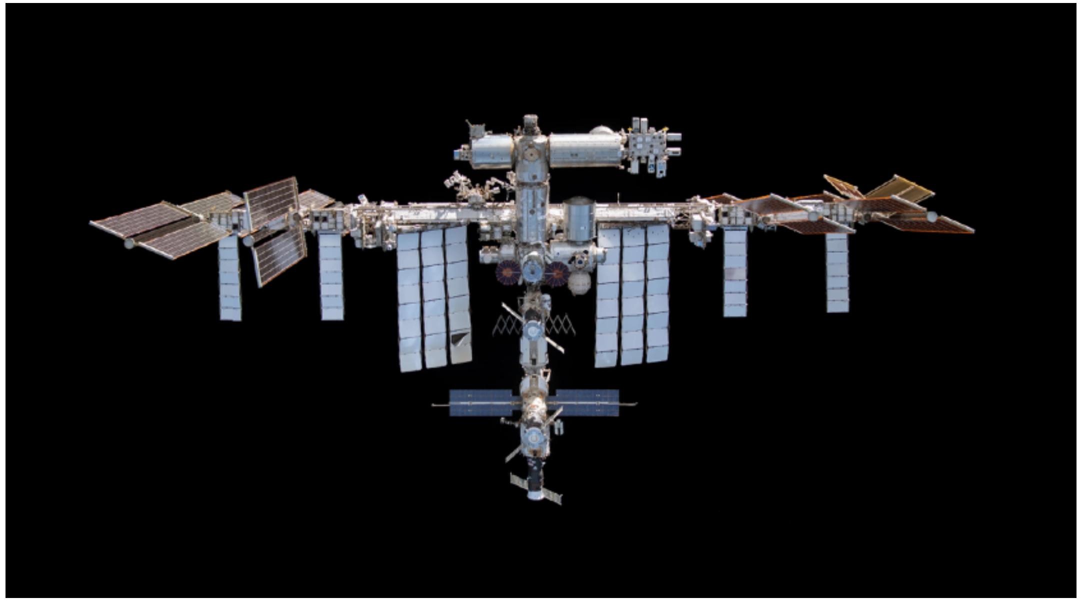
Height above the earth =26599.8 km

Rate $= +45.78\ \mu s/day$

Speed = 13946.3 km/hr

Rate $= -7.21\ \mu s/day$

GPS clocks will tick faster (compared to clocks on the geoid) by $\sim 38.5\ \mu s/day$



ISS, Credit: NASA

Height above the earth = 411.863 km

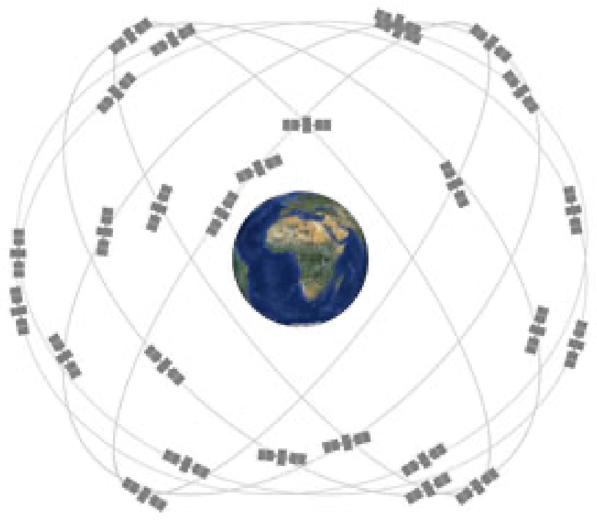
Rate $= +3.78\ \mu s/day$

Speed = 27582.68 km/hr

Rate $= -28.21\ \mu s/day$

ISS clocks will tick slower (compared to clocks on the geoid) by $\sim 24.4\ \mu s/day$

Coordinate time for the Earth: GPS as an example



Credit: GPS

Clocks on the geoid beat at the same rate. If coordinate time is based on clocks at infinity, the proper time of clocks on the geoid is:

$$-L_G = \frac{\Phi_0}{c^2} = -\,6.969290134(0) \times 10^{-10} \sim -\,60.2\ \mu\text{s/day}$$

If we treat Moon like a GPS satellite,

$$\frac{\Delta f}{f} = \frac{3GM_e}{2ac^2} + \frac{\Phi_0}{c^2}\,.$$

static plus centripetal

$a \rightarrow$ semimajor axis

$GM_e \rightarrow$ earth's gravitational parameter

Substitute the Moon's semimajor axis above to obtain a rate offset for the Moon: $58.721\ \mu\text{s/day}$

But there is a problem!
This does not include the effect of the Moon's gravitational potential. Moon's potential should be treated as a tidal potential.

Time on the moon

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A Relativistic Framework to Estimate Clock Rates on the Moon

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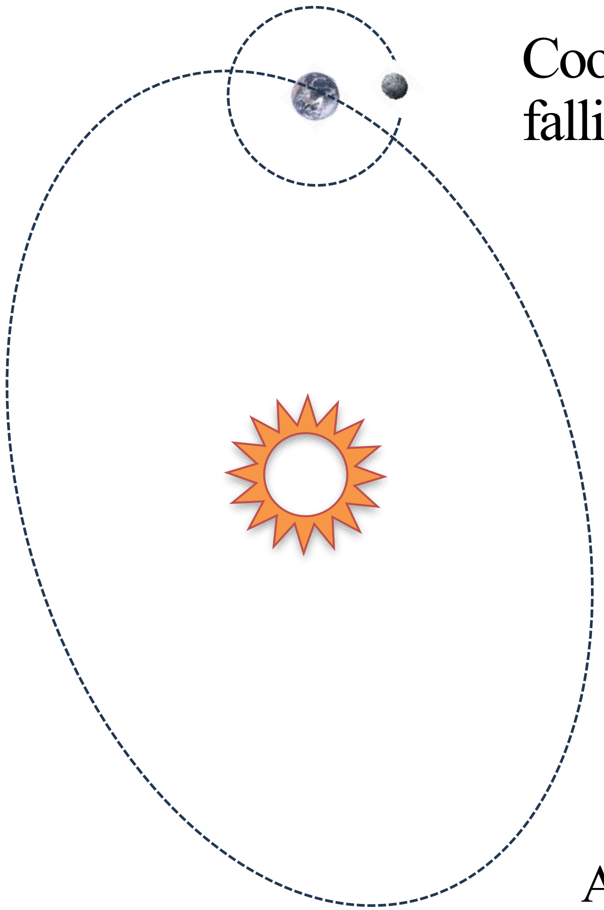
- What is a good choice for the coordinate system that can be used to relate the proper times on the Earth and the Moon?
- What is an appropriate choice for the locations of ideal clocks on the surfaces of the Earth and Moon that makes it easier to compare their proper times?
- What is the proper time difference between clocks on the Moon and the Earth?
- What are the proper time differences between clocks located at the Earth-Moon Lagrange points and the Earth?

Coordinate time: From Earth to Moon



The measure of spacetime is called a metric that is a solution to Einstein's field equations.

Coordinate intervals and proper intervals are connected using the metric, which is appropriate for a freely falling reference frame (**no velocity terms**!).

$$-ds^2 = -\left(1 + \frac{2(\Phi_e - \Phi_0)}{c^2} + \frac{2}{c^2}(\Phi_m + \Phi_s)\Big|_{\text{tidal}}\right)(dx^0)^2 + \left(1 - \frac{2\Phi_e}{c^2} - \frac{2}{c^2}(\Phi_m + \Phi_s)\Big|_{\text{tidal}}\right)(dx^2 + dy^2 + dz^2)$$

Analogous to the earth, clocks on the selenoid beat at the same rate. For the moon, we have:

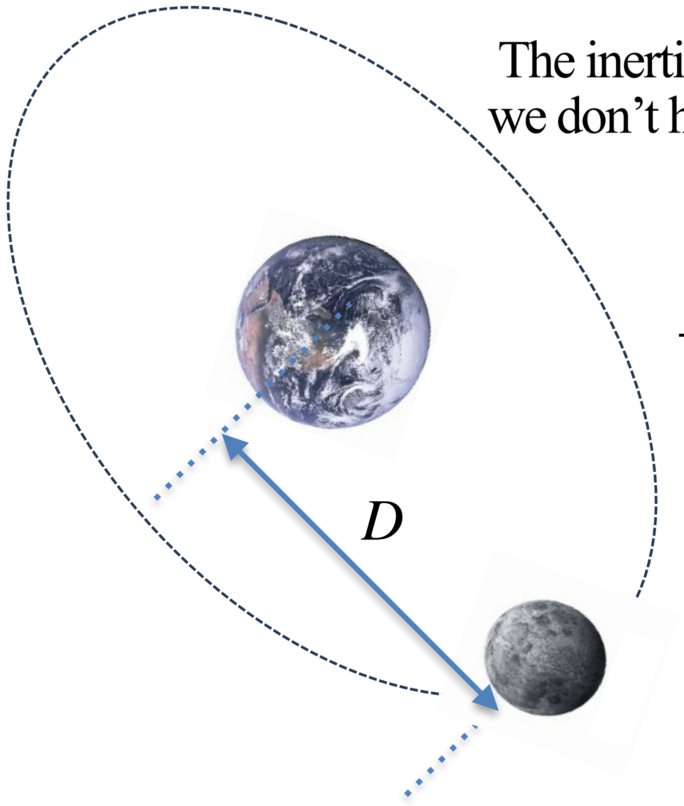
$$L_m = -\frac{\Phi_{0m}}{c^2} = -\left(\frac{\Phi_m\big|_{\theta=\pi/2}}{c^2} - \frac{\omega_m^2 a_m^2}{2c^2}\right) = 3.13881(15) \times 10^{-11}$$

Apart from tidal effects, standard clocks at rest on an effective equipotential of the rotating Moon will beat at equal rates and can be used to define the rate of coordinate time on the Moon.

Center of mass for the Earth-Moon system

The Earth and the Moon orbit around their mutual center of mass in different Keplerian orbits. The center of mass of the Earth-Moon system orbits around the Sun in an approximately Keplerian orbit. A fictitious locally freely-falling inertial frame is introduced at the Earth-Moon center of mass.

$$-ds^2 = -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2}\right)(dx^0)^2 + \left(1 - \frac{2\Phi_e}{c^2} - \frac{2\Phi_m}{c^2}\right)(dx^2 + dy^2 + dz^2)$$



The inertial frame is centered on the Earth-Moon center of mass with a well-defined coordinate time. But we don't have to know what it is. Proper time is the time elapsed on the clocks at the geoid (equator).

$$-c^2 d\tau_e^2 = -\left(1 + \frac{2\Phi_e}{c^2}\bigg|_{R_{Eeq}} + \frac{2\Phi_m}{c^2}\bigg|_{R_{Eeq}}\right)(dx^0)^2 + \frac{(\mathbf{V_e} + \mathbf{v_e})^2}{c^2}(dx^0)^2$$

Earth's velocity

clock velocity (not fixed)

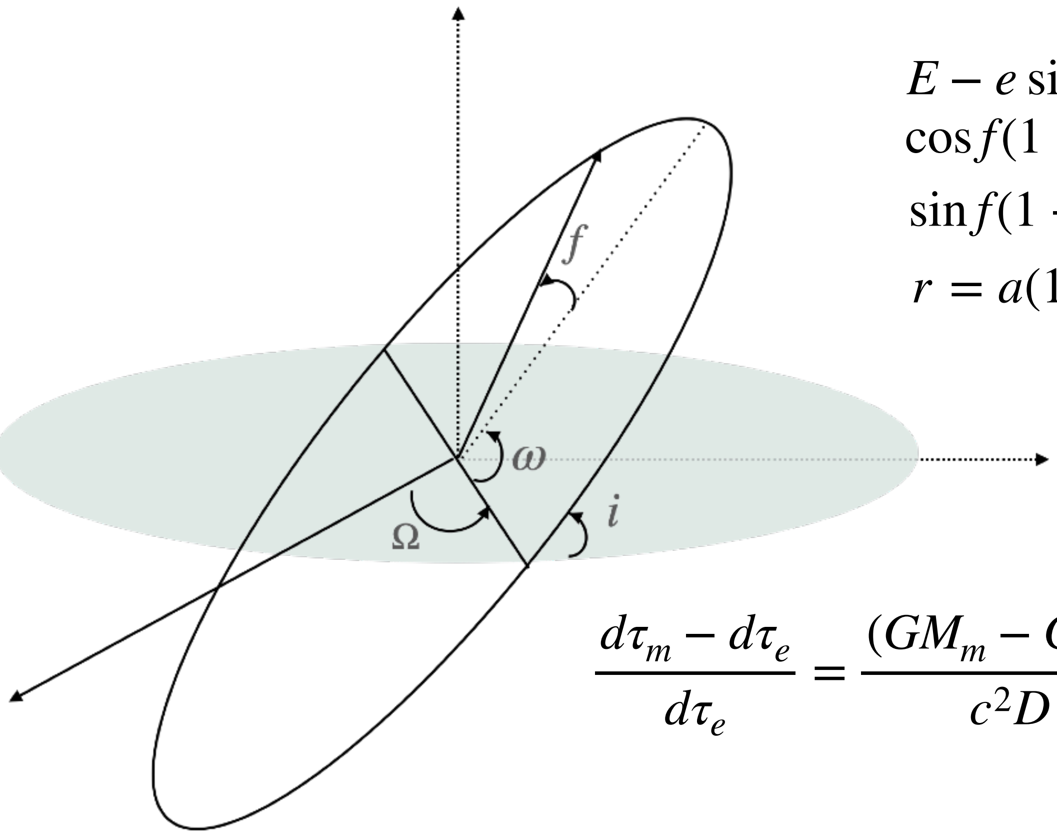
$$\frac{\Phi_m}{c^2}\bigg|_{R_{Eeq}} = -\frac{GM_m}{c^2 D}$$

definition of selenoid

$$\frac{d\tau_m - d\tau_e}{d\tau_e} = \frac{(GM_m - GM_e)}{c^2 D} + \frac{\Phi_{0m} - \Phi_0}{c^2} - \frac{1}{2c^2}(V_m^2 - V_e^2)$$

definition of geoid

Center of mass for the earth-moon: Keplerian orbits



$$E - e\sin E = (t - t_P)\sqrt{GM}a^{-3/2} = n(t - t_P)$$
$$\cos f(1 - e\cos E) = \cos E - e$$
$$\sin f(1 - e\cos E) = \sin E\sqrt{1 - e^2}$$
$$r = a(1 - e\cos E)$$

$E \rightarrow$ eccentric anomaly
$f \rightarrow$ true anomaly
$a \rightarrow$ semimajor axis
$e \rightarrow$ eccentricity
$t_P \rightarrow$ time of perigee passage

$$\frac{GM_m}{GM_e + GM_m} = \frac{M_m}{M_T} \sim 0.01215$$

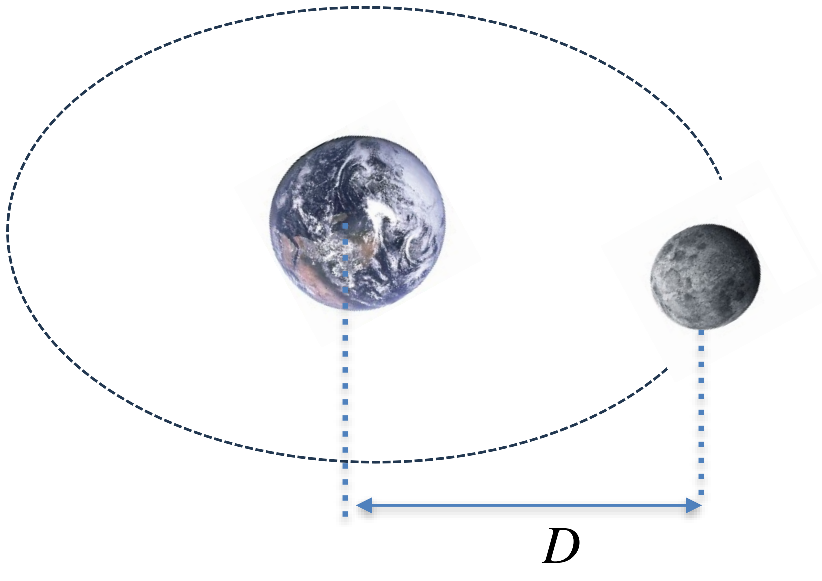
$$\frac{d\tau_m - d\tau_e}{d\tau_e} = \frac{(GM_m - GM_e)}{c^2 D} + \frac{\Phi_{0m} - \Phi_0}{c^2} - (1 - 2\mu)\left(\frac{GM_T}{2ac^2}\right)\left(\frac{1 + 2e\cos(f) + e^2}{1 - e^2}\right)$$

The center of mass system coordinate time has dropped out. The proper time of clocks on Earth can be directly referenced to the proper time on the Moon. The rate offset between the Earth and the Moon can be estimated as a function of true anomaly.

Clock rate offsets for the Moon and earth-Moon Lagrange points

$$\frac{d\tau_m - d\tau_e}{d\tau_e} = 6.48378(15) \times 10^{-10} - 1.25502518(89) \times 10^{-12} \cos(f)$$

$$= 56.0199(12) - 0.10843417(89)\cos(f)\ \mu s/\text{day}$$



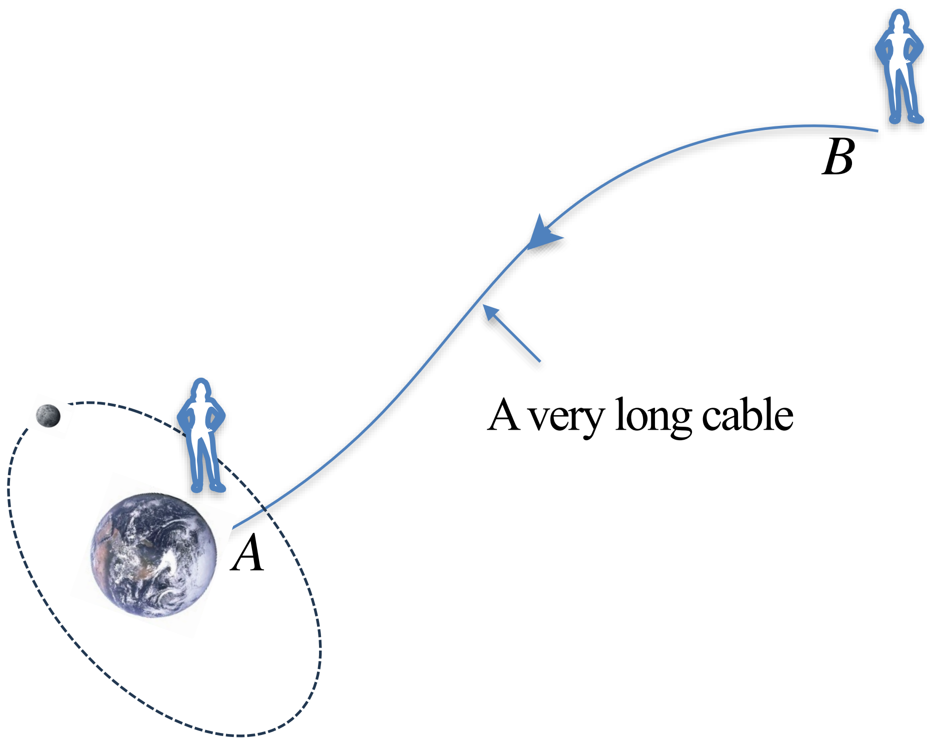
A clock near the Moon's equator ticks faster than one near the Earth's equator, accumulating an extra 56.02 microseconds per day over the duration of a lunar orbit.

quantity	location	rate (μs/day)
$(d\tau_m/d\tau_e) - 1$	lunar surface	$56.0199(12) - 0.10843417(89)\cos(f)$
$(d\tau_{L_1}/d\tau_e) - 1$	L_1	$58.612420(12) - 0.10736106(12)\cos(f)$
$(d\tau_{L_2}/d\tau_e) - 1$	L_2	$58.619639(12) - 0.12445590(12)\cos(f)$
$(d\tau_{L_4}/d\tau_e) - 1$	L_4/L_5	$58.707278(12) - 0.11045150(89)\cos(f)$

Table 3. Clock rates computed for various points of interest.

Clock comparison in GR

Thought experiment: A tale of two identical clocks at different locations (mind the cable!)



Send clock B signal through a cable that is not allowed to move during this experiment.

What is a clock?

A device that locks to a frequency source and tracks it based on a definition. In the simplest form: Count cycles. When a fixed number is reached, increment the counter by a unit and repeat.

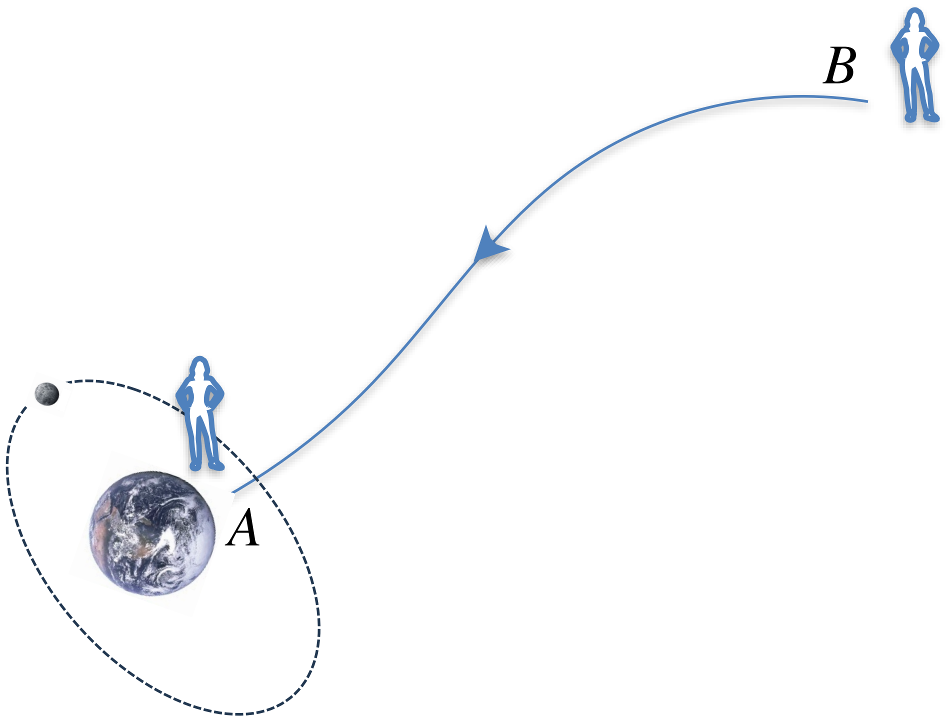
Proper frequency (and time) depend on the gravitational potential at the location of the clock and the motion of the clock in space.

Since there are no long cables connecting every point in spacetime, we have to establish a coordinate time locally that is operationally convenient.

Coordinate frequency is conserved. That is because the separation between successive waveforms or pulses constructed with such signals stays fixed (in spacetime) if you don't move the cable.

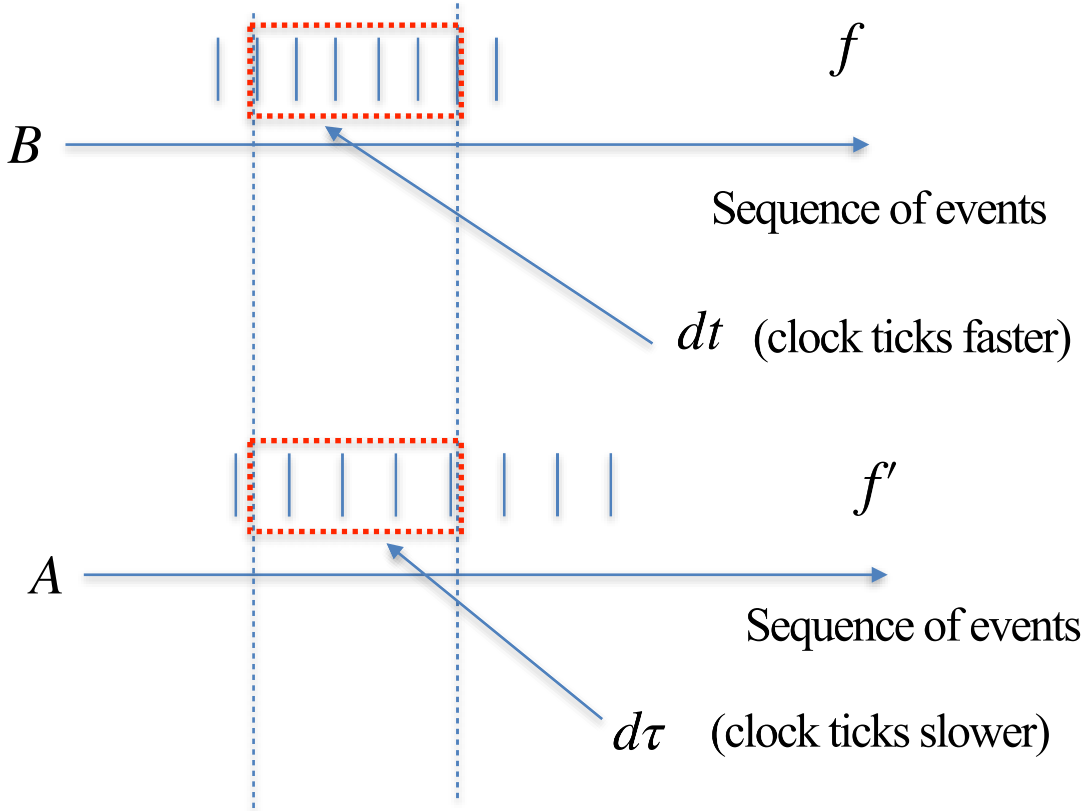
Clock comparison in GR

Thought experiment: A tale of two identical clocks at different locations (mind the cable!)



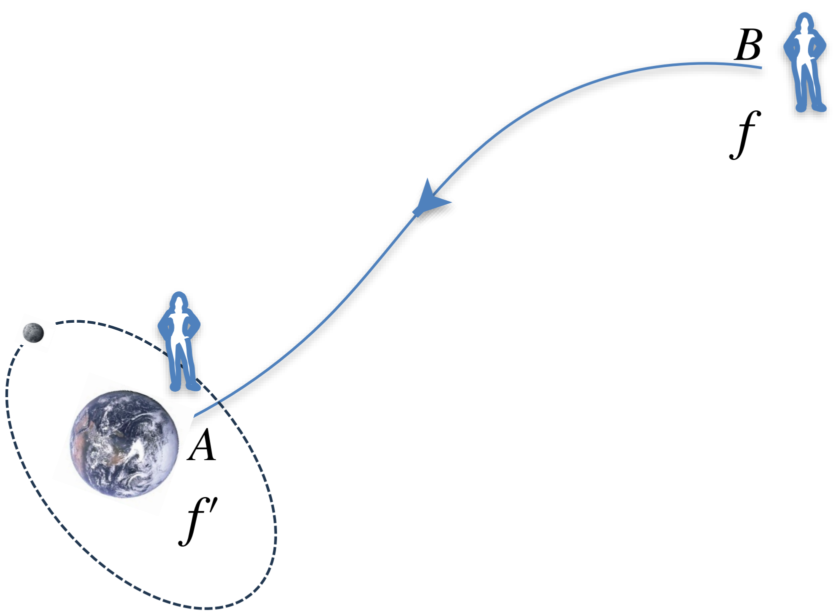
There is a distinction between transporting the clock versus transporting the clock signal—relativistically speaking!

Pulses generated by clocks A and B using their proper frequencies can be compared at the location of clock A.



Clock comparison in GR

Thought experiment: A tale of two identical clocks at different locations (mind the cable!)



$$\frac{d\tau}{dt} - 1 = \frac{f' - f}{f}$$

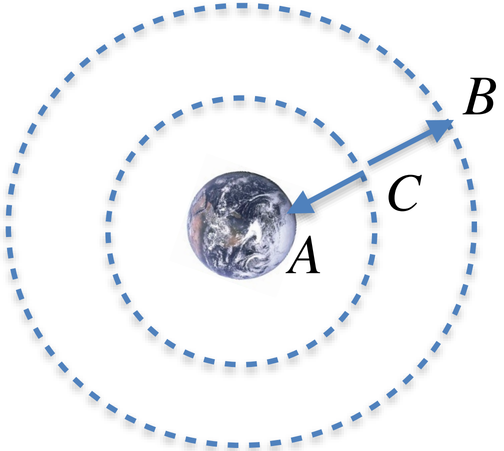
$f' \rightarrow$ measured frequency

$f \rightarrow$ coordinate frequency

$d\tau \rightarrow$ proper time interval

$dt \rightarrow$ realizes coordinate time interval if B is moved to infinity (sufficiently far away away from any gravitational field)

This is the rate offset that is of interest to us for the calculations.



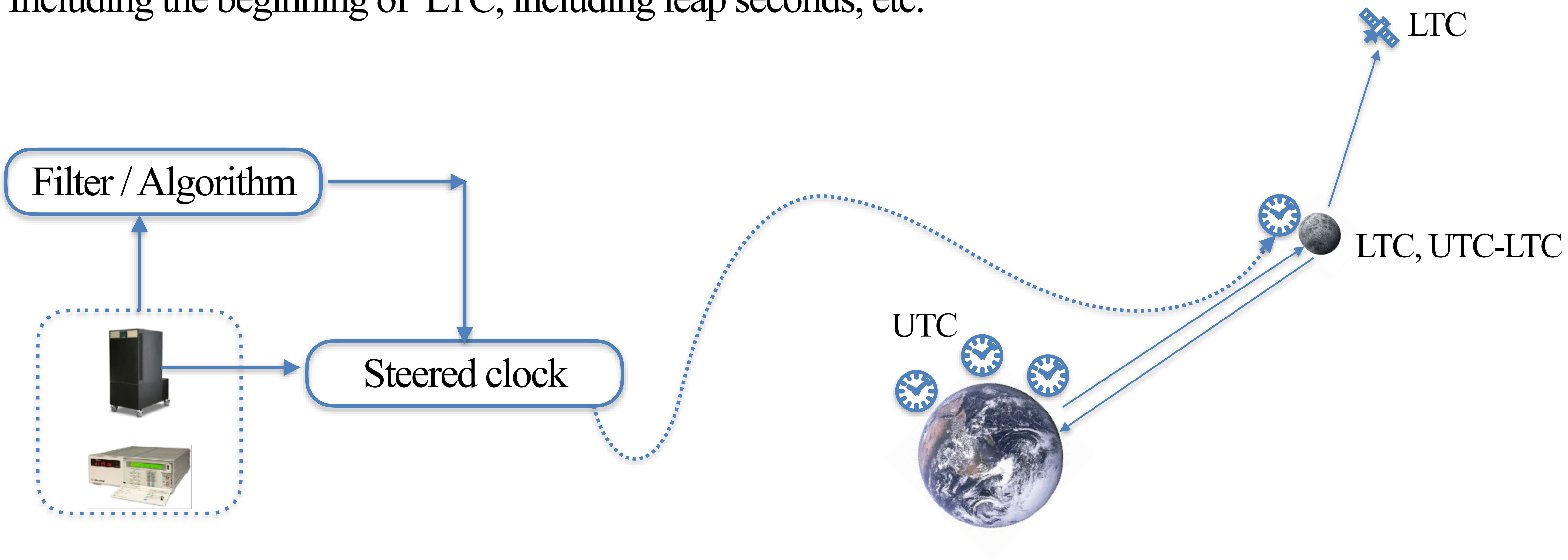
To match the measured frequency of clock C with clock B, apply positive correction to C

To match the measured frequency of clock C with clock A, apply negative correction to C

Realizing Lunar Coordinated Time (LTC)

Various aspects of an elementary realization of LTC are still under discussion.

Including the beginning of LTC, including leap seconds, etc.



Concluding remarks

Terms corresponding to the potentials of the Earth and Moon and definitions of the geoid and selenoid: $-\frac{GM_e}{Dc^2} - \frac{\phi_0}{c^2}, \quad -\frac{GM_m}{Dc^2} - \frac{\phi_{0m}}{c^2}$

Terms corresponding to the speeds of the Earth and Moon w.r.t. the Earth-Moon center of mass frame: $-\frac{V_e^2}{2c^2} + \frac{V_m^2}{2c^2}$

If the appropriate terms are not accounted correctly, results in incorret estimates of rate offsets: $\sim 58.7\ \mu s$, $\sim 57.0\ \mu s$, $\sim 56.5\ \mu s$

Tidal corrections are of the order of : ~ less than 10 ns.

Corrections due to Lorentz contraction: ~ less than 10 ps.

Concluding remarks

A freely falling coordinate system may be used to accurately estimate the Moon's rate offsets from the Earth's geoid.

Using the above, establishing a coordinate time for the Moon is very similar to establishing a coordinate time (also known as GPS time) for the Earth.

This framework can be extended to compute rate offsets of other celestial bodies in the solar system and Lagrange points (cislunar space).

Efforts to establish a reference frame and a reference time are continuing.

Acknowledgment

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Link to the paper: https://iopscience.iop.org/article/10.3847/1538-3881/ad643a

Lagrange points and cislunar space (extra)

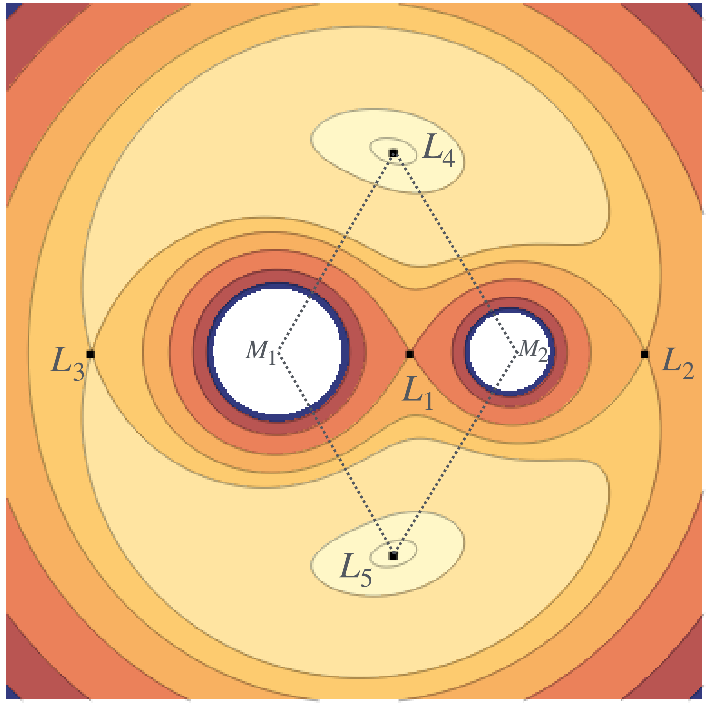
Restricted three-body problem solution by Giuseppe Lodovico Lagrangia in 1772

$$m \ll M_e \quad \text{and} \quad m \ll M_m$$

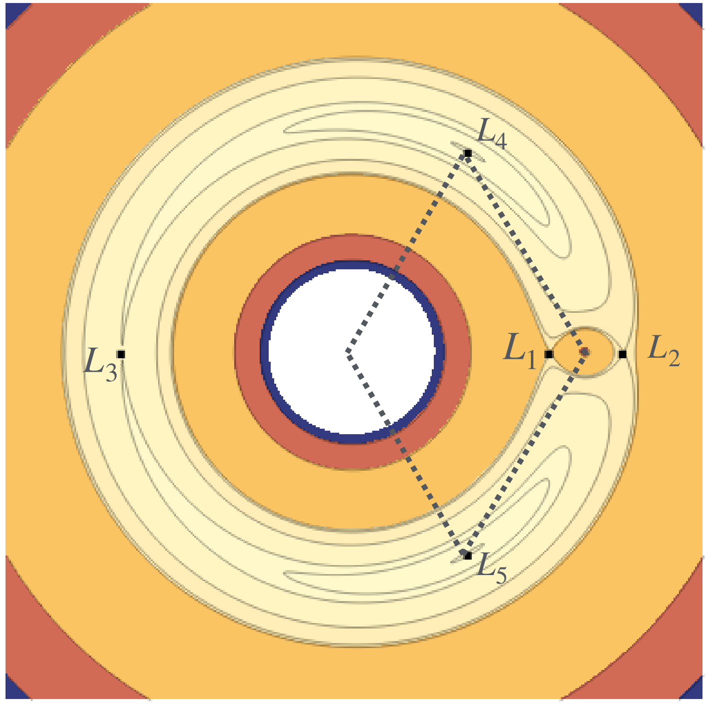
$$\vec{F} = -GM_e \frac{\vec{r} - \vec{r}_e}{|\vec{r} - \vec{r}_e|^3} - GM_m \frac{\vec{r} - \vec{r}_m}{|\vec{r} - \vec{r}_m|^3}$$

$$\vec{F}_m = \vec{F} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \frac{d\vec{r}}{dt}$$

centrifugal — coriolis



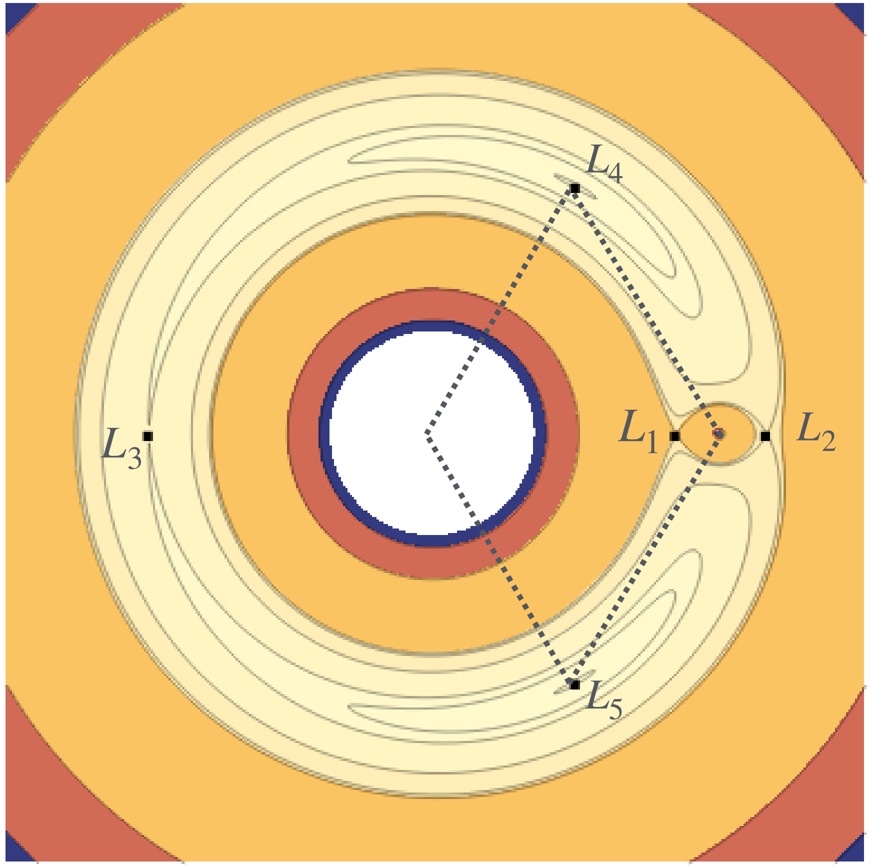
$M_2/M_1 = 0.5$ (just an example)



$M_m/M_e = 0.0123$ (actual)

Lagrange points and cislunar space (extra)

Stable Lagrange points L_4 and L_5—in this case—are also Trojan points. The coriolis acceleration provides stability for these orbits. At all other Lagrange points, spacecraft will drift away in ~10 (earth) days.



$M_m/M_e = 0.0123$ (actual)

$$\frac{d\tau_{L1} - d\tau_e}{d\tau_e} = -\frac{GM_e}{c^2 D(1-x_1)} - \frac{GM_m}{c^2 D x_1} - \frac{\Phi_0}{c^2} - \frac{GM_T(1-\mu-x_1)^2}{2c^2 a(1-e^2)}(1+2e\cos(f)+e^2)$$

$$+\frac{GM_m}{c^2 D} + \mu^2 \frac{GM_T}{2c^2 a(1-e^2)}(1+2e\cos(f)+e^2)$$

$Dx_1 \rightarrow$ Distance from Moon's center to L_1

For the stable Trojan points, the result is:

$$\frac{d\tau_{L4} - d\tau_e}{d\tau_e} = -\frac{GM_T}{c^2 D} - \frac{\Phi_0}{c^2} + \frac{GM_m}{c^2 D} - \frac{GM_T}{2ac^2(1-e^2)}\left(1-\mu^2\right)(1+2e\cos(f)+e^2)$$